INFINITY '09

An Inverse Method for Markov Decision Processes

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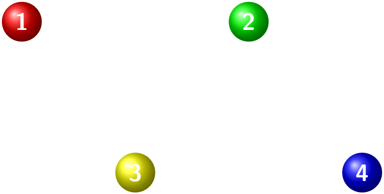
Context: Hardware Verification

- Verification of real time systems with stochastic behavior
  - Need to express probabilities
  - Need to express infinite behaviors
  - Use of Markov decision processes [Bel57, How60]
- Need for adjusting some timings or costs of the system
  - Use of parameters (unknown constants)
  - Definition of a zone of good behavior for the parameters

Markov Decision Process (MDP)

- Weighted labeled directed graph augmented with probabilities
  - A set of states $S = \{s_1, \ldots, s_n\}$



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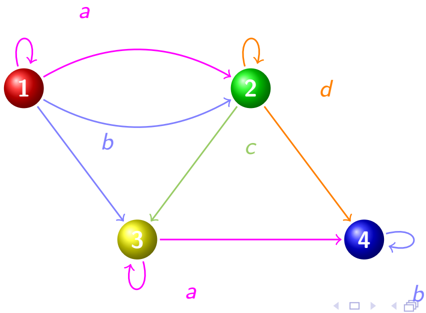

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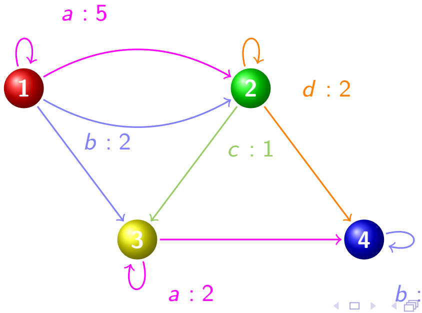

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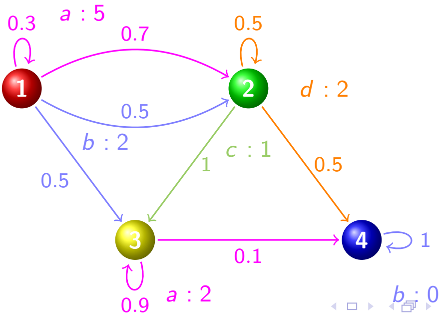

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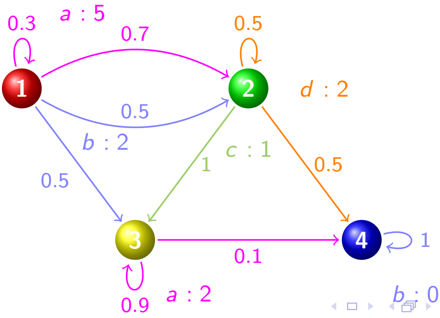

The Direct Problem: Optimal Policy

- Policy μ: function from states to actions $S \rightarrow A$
  - Resolves the non-determinism
  - The MDP becomes a Markov Chain [KMST59]
- Optimal policy: policy such that the sum of the weights until the absorbing state is minimal

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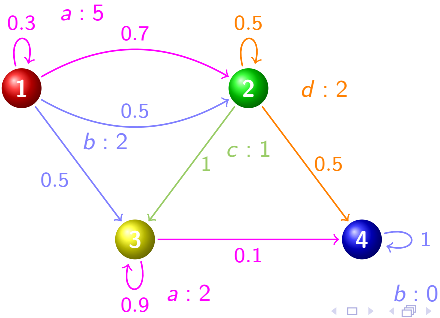

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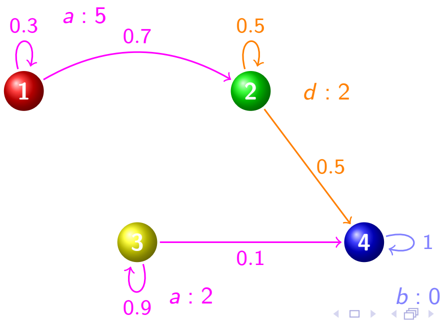
  - $\mu = \{1 \rightarrow a, \quad 2 \rightarrow d, \quad 3 \rightarrow a\}$



The Inverse Problem

- The direct problem
  - Given an MDP, compute an optimal policy
- The inverse problem
  - Given an MDP and an optimal policy, can we change the values of some weights so that this policy remains optimal?

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  - Given an MDP, compute an optimal policy
- The inverse problem
  - Given an MDP and an optimal policy, can we change the values of some weights so that this policy remains optimal?
- More formally...

Goal

Given an MDP $\mathcal{M}$ and an optimal policy μ_0, compute a constraint K_0 on the weights seen as parameters such that, for any value of the parameters, the policy μ_0 remains optimal

Outline

Outline

The Classical Value Determination Algorithm

- Used by the policy iteration algorithm to compute the optimal policy
- Inputs
  - A Markov decision process $\mathcal{M} = (S, A, Prob, w)$
  - A policy μ
- Output
  - A value function v, associating a value to every state s, i.e., the cost from s to the absorbing state in $\mathcal{M}$ restricted to policy μ

Algorithm (Value Determination)

SOLVE $\{v(s) = w(s, \mu[s]) + \sum_{s' \in S} Prob(s, \mu[s], s') \times v(s')\}_{s \in S \setminus s_n}$

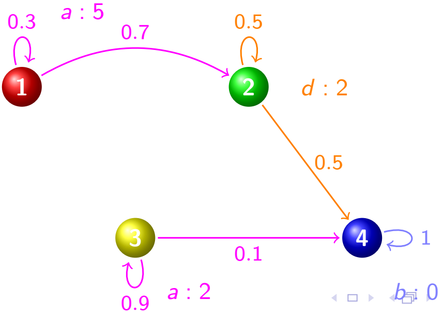
The Classical Value Determination Algorithm: Application

Algorithm (Value Determination)

SOLVE $\{v(s) = w(s, \mu[s]) + \sum_{s' \in S} Prob(s, \mu[s], s') \times v(s')\}_{s \in S \setminus s_n}$

$\mu = \{1 \rightarrow a, \quad 2 \rightarrow d, \quad 3 \rightarrow a\}$

$$v(1) = w(1, a) + 0.3 \times v(1) + 0.7 \times v(2)$$
$$v(2) = w(2, d) + 0.5 \times v(2) + 0.5 \times v(4)$$
$$v(3) = w(3, a) + 0.9 \times v(3) + 0.1 \times v(4)$$
$$v(4) = 0$$



The Classical Value Determination Algorithm: Application

Algorithm (Value Determination)

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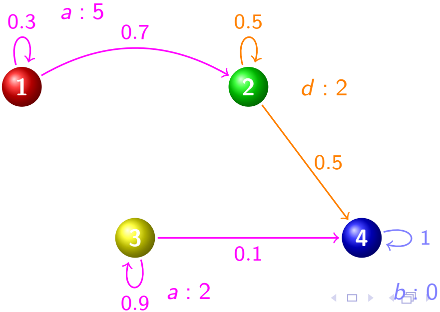
$\mu = \{1 \to a, \quad 2 \to d, \quad 3 \to a\}$

$$
\begin{aligned}
v(1) &= w(1,a) + 0.3 \times v(1) + 0.7 \times v(2) & \\
v(2) &= w(2,d) + 0.5 \times v(2) + 0.5 \times v(4) & = 4 \\
v(3) &= w(3,a) + 0.9 \times v(3) + 0.1 \times v(4) & = 20 \\
v(4) &= 0 &
\end{aligned}
$$



The Classical Value Determination Algorithm: Application

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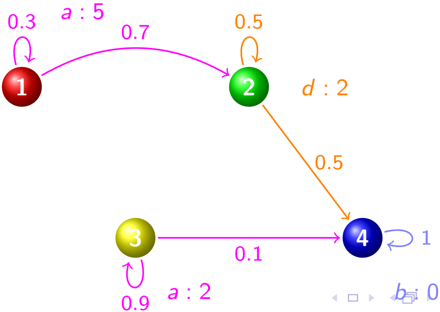
$\mu = \{1 \rightarrow a, \quad 2 \rightarrow d, \quad 3 \rightarrow a\}$

$$\begin{aligned} v(1) &= w(1, a) + 0.3 \times v(1) + 0.7 \times v(2) &&= \frac{78}{7} \\ v(2) &= w(2, d) + 0.5 \times v(2) + 0.5 \times v(4) &&= 4 \\ v(3) &= w(3, a) + 0.9 \times v(3) + 0.1 \times v(4) &&= 20 \\ v(4) &= 0 \end{aligned}$$



The Classical Policy Iteration Algorithm

- Input: A Markov decision process $\mathcal{M} = (S, A, Prob, w)$
- Output: An optimal policy μ
- Principle:
  1. Start with a random policy
  2. Compute the value function, using algorithm *ValueDet*
  3. Choose a strictly better policy, and go to (2) until fixpoint

Algorithm (Policy Iteration)

REPEAT UNTIL FIXPOINT

$v := ValueDet(M, \mu)$

for each $s \in S \setminus s_n$ **DO**

$optimum := v[s]$

for each $a \in e(s)$ **DO**

IF $w(s, a) + \sum_{s' \in S} Prob(s, a, s')v(s') < optimum$ **THEN**

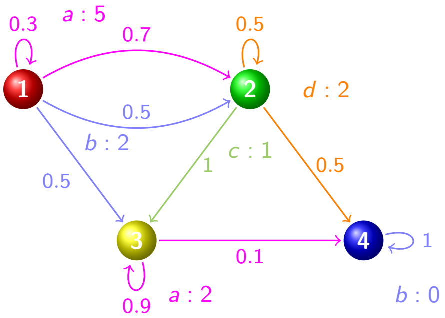
$optimum := w(s, a) + \sum_{s' \in S} Prob(s, a, s')v(s')$

$\mu[s] := a$

The Classical Policy Iteration Algorithm: Application

$v(1) = \frac{197}{7}$
$v(2) = 21$
$v(3) = 20$
$v(4) = 0$

μ :
$1 \rightarrow a$
$2 \rightarrow c$
$3 \rightarrow a$

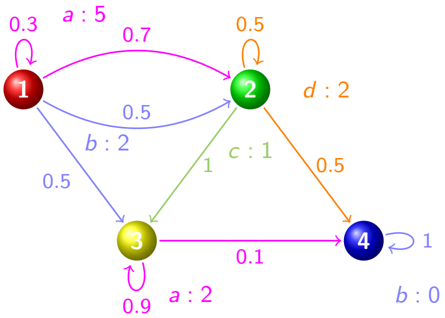


1. We start from an arbitrary policy

The Classical Policy Iteration Algorithm: Application

$v(1) = 14$
$v(2) = 4$
$v(3) = 20$
$v(4) = 0$

μ :
$1 \to b$
$2 \to d$
$3 \to a$



1. We start from an arbitrary policy
2. We improve policy for states 1 and 2

The Classical Policy Iteration Algorithm: Application

$v(1) = \frac{78}{7}$
$v(2) = 4$
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μ :
$1 \to a$
$2 \to d$
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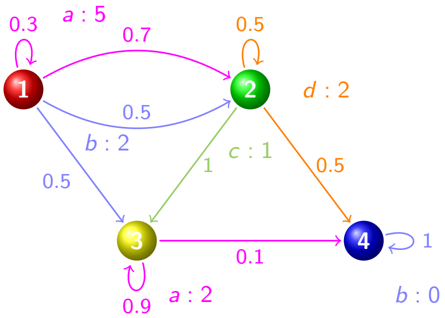

1. We start from an arbitrary policy
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$v(1) = \frac{78}{7}$
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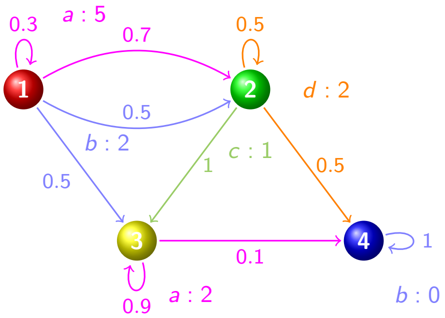
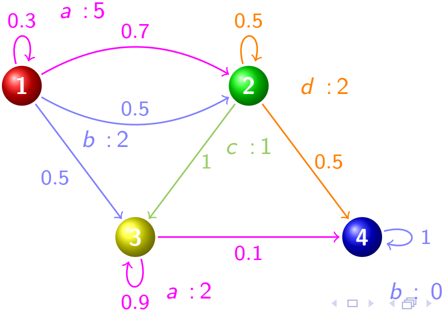


1. We start from an arbitrary policy
2. We improve policy for states 1 and 2
3. We improve policy for state 1
4. Fixpoint is reached: the policy μ is optimal for $\mathcal{M}$

Outline

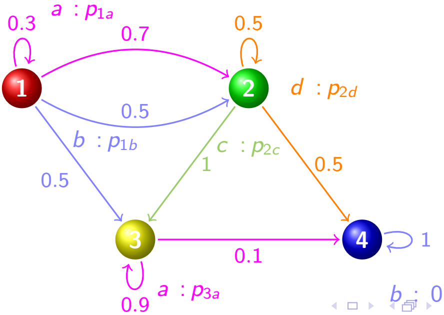
Markov Decision Process

- Markov Decision Process
  - A set of states $S = \{s_1, \ldots, s_n\}$, including one absorbing state s_n
  - A set A of actions
  - A weight function w, associating a cost $w(s, a)$ to every state s and action a
  - A probability function $Prob$, associating a probability to every edge, such that the sum of the probabilities of leaving a state s through action a is equal to 1, i.e., $\sum_{s' \in S} Prob(s, a, s') = 1$



Parametric Markov Decision Process

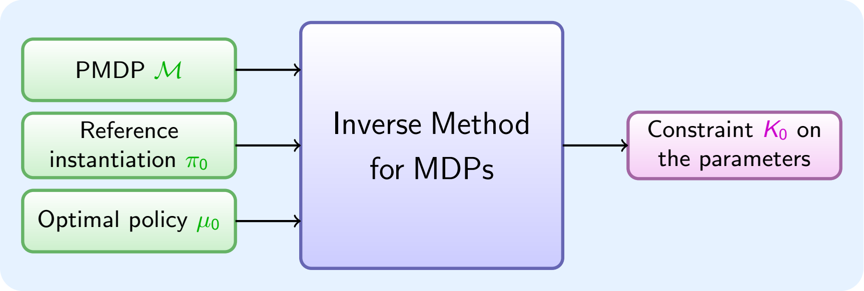
- Markov Decision Process with parametric weights
  - A set of states $S = \{s_1, \ldots, s_n\}$, including one absorbing state s_n
  - A set A of actions
  - A parametric weight function W, associating a parametric cost (i.e., unknown constant) $W(s, a)$ to every state s and action a
  - A probability function $Prob$, associating a probability to every edge, such that the sum of the probabilities of leaving a state s through action a is equal to 1, i.e., $\sum_{s' \in S} Prob(s, a, s') = 1$



Parametric Markov Decision Process: Remarks

- Instantiating a PMDP $\mathcal{M}$ with a valuation π of the parameters gives a (non-parametric) MDP
  - Denoted by $\mathcal{M}[\pi]$
- A PMDP models the behavior of an infinite number of MDPs
- The parametrization of an MDP into a PMDP is similar to the parametrization of a Timed Automaton into a Parametric Timed Automaton

Inputs and Outputs (1/2)

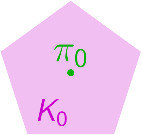


Inputs and Outputs (2/2)

- Inputs
  - A Parametric MDP $\mathcal{M}$
  - A reference instantiation π_0 of all the parameters of $\mathcal{M}$
  - A policy μ_0 optimal for $\mathcal{M}[\pi_0]$

Inputs and Outputs (2/2)

- Inputs
  - A Parametric MDP $\mathcal{M}$
  - A reference instantiation π_0 of all the parameters of $\mathcal{M}$
  - A policy μ_0 optimal for $\mathcal{M}[\pi_0]$
- Output: generalization
  - A constraint K_0 on the parameters such that
    - ⋆ $\pi_0 \models K_0$
    - ⋆ The policy μ_0 is optimal for $\mathcal{M}[\pi]$, for all $\pi \models K_0$



The General Idea

Given a PMDP $\mathcal{M}$, an instantiation π_0 of the parameters, and a policy μ_0 optimal for $\mathcal{M}[\pi_0]$

1. Compute a parametric value function for $\mathcal{M}$ and μ_0, using a parametric version of the value determination algorithm
2. Generate constraints on the parameters of $\mathcal{M}$, using a parametric version of the policy iteration algorithm

The Parametric Value Determination Algorithm

- Straightforward adaptation of the value determination algorithm to the parametric case
- Inputs
  - A parametric Markov decision process $\mathcal{M} = (S, A, Prob, W)$
  - A policy μ
- Output
  - A parametric value function V, associating a parametric value to every state s, i.e., the parametric cost from s to the absorbing state in $\mathcal{M}$ restricted to policy μ

Algorithm (Parametric Value Determination *P-ValueDet*)

SOLVE $\{V(s) = W(s, \mu[s]) + \sum_{s' \in S} Prob(s, \mu[s], s') \times V(s')\}_{s \in S \setminus s_n}$

Algorithm *P-ValueDet*: Application

Algorithm (Parametric Value Determination *P-ValueDet*)

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$\mu = \{1 \rightarrow a, \;\; 2 \rightarrow d, \;\; 3 \rightarrow a\}$

$$V(1) = W(1, a) + 0.3 \times V(1) + 0.7 \times V(2)$$
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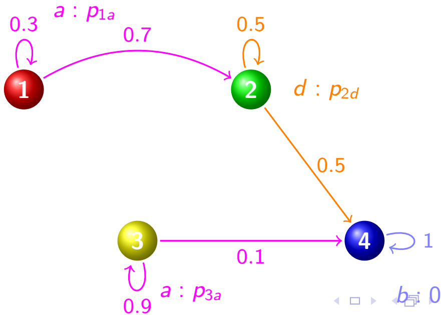
$$V(4) = 0$$



Algorithm *P-ValueDet*: Application

Algorithm (Parametric Value Determination *P-ValueDet*)

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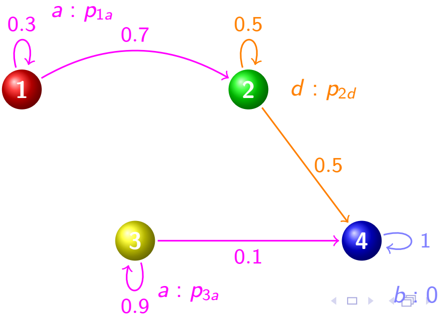
$\mu = \{1 \to a, \quad 2 \to d, \quad 3 \to a\}$

$$\begin{aligned}
V(1) &= W(1,a) + 0.3 \times V(1) + 0.7 \times V(2) &&= \tfrac{10}{7} \times p_{1a} + 2 \times p_{2d} \\
V(2) &= W(2,d) + 0.5 \times V(2) + 0.5 \times V(4) &&= 2 \times p_{2d} \\
V(3) &= W(3,a) + 0.9 \times V(3) + 0.1 \times V(4) &&= 10 \times p_{3a} \\
V(4) &= 0
\end{aligned}$$



The Algorithm *InverseMethod*

- Inputs
  - A PMDP $\mathcal{M} = (S, A, Prob, W)$
  - An instantiation π_0 of the parameters
  - A policy μ_0 optimal for $\mathcal{M}[\pi_0]$
- Output
  - A constraint K_0 on the parameters solving the inverse problem
- Principle
  - For each state s, for each action a, generate an inequality stating that the optimal policy $\mu_0[s]$ is better than a for s

Algorithm (*InverseMethod*)

$V \ := P\text{-}ValueDet(M, \mu_0)$

$K_0 := True$

FOR EACH $s \in S \setminus \{s_n\}$ **DO**

  FOR EACH $a \in e(s)$ $s.t.$ $a \neq \mu_0[s]$ **DO**

    $K_0 := K_0 \wedge \{W(s,a) + \sum_{s' \in S} Prob(s,a,s')V[s'] \geq V[s]\}$

Properties of the Algorithm *InverseMethod*

Theorem (Correctness)

Given a PMDP $\mathcal{M}$, a reference instantiation π_0 and a policy μ_0 optimal for $\mathcal{M}[\pi_0]$, the constraint K_0 output by the algorithm InverseMethod is such that

- $\pi_0 \models K_0$, *and*
- μ_0 *is optimal for* $\mathcal{M}[\pi]$, *for all* $\pi \models K_0$

Theorem (Termination and complexity)

The algorithm InverseMethod terminates in polynomial time.

Application to Our Example

π_0 :

$p_{1a} = 5 \qquad p_{1b} = 2$

$p_{2c} = 1 \qquad p_{2d} = 2$

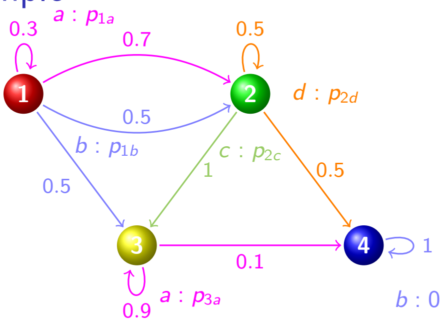
$p_{3a} = 2$

μ_0 :

$1 \rightarrow a$

$2 \rightarrow d$

$3 \rightarrow a$



Application to Our Example

π_0 :

$p_{1a} = 5 \qquad p_{1b} = 2$

$p_{2c} = 1 \qquad p_{2d} = 2$

$p_{3a} = 2$

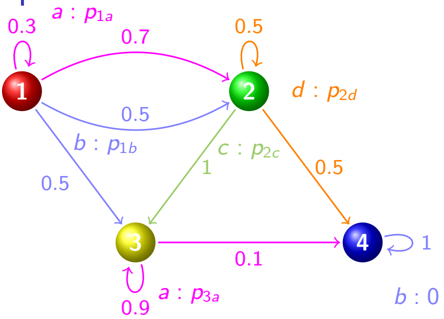
μ_0 :

$1 \rightarrow a$

$2 \rightarrow d$

$3 \rightarrow a$

$$V(1) = \frac{10}{7} \times p_{1a} + 2 \times p_{2d}$$
$$V(2) = 2 \times p_{2d}$$
$$V(3) = 10 \times p_{3a}$$



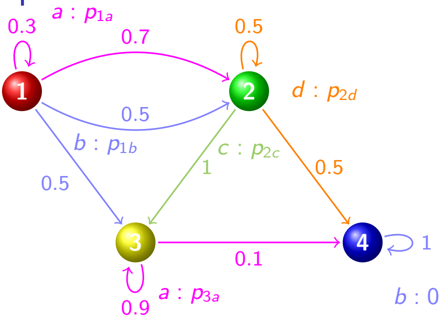
Application to Our Example

π_0 :

$p_{1a} = 5$ $\quad p_{1b} = 2$
$p_{2c} = 1$ $\quad p_{2d} = 2$
$p_{3a} = 2$

μ_0 :

$1 \rightarrow a$
$2 \rightarrow d$
$3 \rightarrow a$



$$V(1) = \frac{10}{7} \times p_{1a} + 2 \times p_{2d}$$
$$V(2) = 2 \times p_{2d}$$
$$V(3) = 10 \times p_{3a}$$

$K_0 = True$

Application to Our Example

π_0 :

$p_{1a} = 5 \qquad p_{1b} = 2$

$p_{2c} = 1 \qquad p_{2d} = 2$

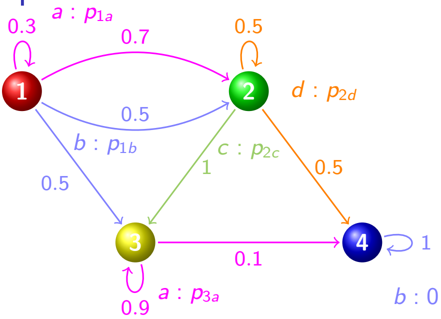
$p_{3a} = 2$

μ_0 :

$1 \rightarrow a$

$2 \rightarrow d$

$3 \rightarrow a$



$$V(1) = \frac{10}{7} \times p_{1a} + 2 \times p_{2d}$$
$$V(2) = 2 \times p_{2d}$$
$$V(3) = 10 \times p_{3a}$$

$K_0 =$

$p_{1b} + \frac{1}{2}V(2) + \frac{1}{2}V(3) \geq V(1)$ %% for 1 and b

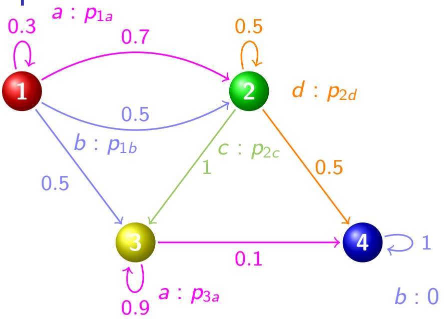
Application to Our Example

π_0 :

$p_{1a} = 5 \qquad p_{1b} = 2$

$p_{2c} = 1 \qquad p_{2d} = 2$

$p_{3a} = 2$



μ_0 :

$1 \rightarrow a$

$2 \rightarrow d$

$3 \rightarrow a$

$$V(1) = \frac{10}{7} \times p_{1a} + 2 \times p_{2d}$$
$$V(2) = 2 \times p_{2d}$$
$$V(3) = 10 \times p_{3a}$$

$K_0 =$

$p_{1b} + \frac{1}{2}V(2) + \frac{1}{2}V(3) \geq V(1)$ %% for 1 and b

$\wedge \quad p_{2c} + V(3) \geq V(2)$ %% for 2 and c

Application to Our Example

π_0 :

$p_{1a} = 5 \qquad p_{1b} = 2$

$p_{2c} = 1 \qquad p_{2d} = 2$

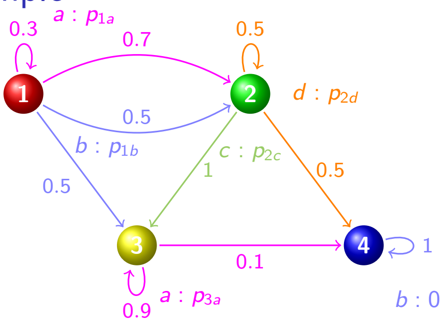
$p_{3a} = 2$

μ_0 :

$1 \rightarrow a$

$2 \rightarrow d$

$3 \rightarrow a$



$$V(1) = \frac{10}{7} \times p_{1a} + 2 \times p_{2d}$$
$$V(2) = 2 \times p_{2d}$$
$$V(3) = 10 \times p_{3a}$$

$K_0 =$

$$p_{1b} + 5p_{3a} \geq \frac{10}{7} p_{1a} + p_{2d}$$
$$\wedge \quad p_{2c} + 10p_{3a} \geq 2p_{2d}$$

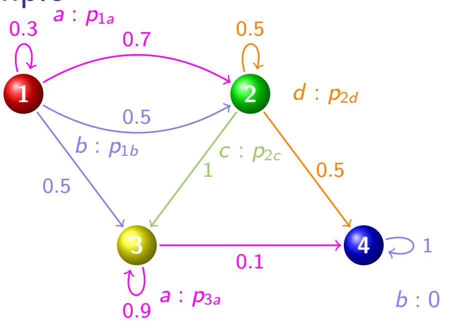
Application to Our Example

π_0 :

$p_{1a} = 5 \qquad p_{1b} = 2$

$p_{2c} = 1 \qquad p_{2d} = 2$

$p_{3a} = 2$



μ_0 :

$1 \to a$

$2 \to d$

$3 \to a$

$$V(1) = \frac{10}{7} \times p_{1a} + 2 \times p_{2d}$$
$$V(2) = 2 \times p_{2d}$$
$$V(3) = 10 \times p_{3a}$$

$$K_0 = \quad p_{1b} + 5p_{3a} \geq \frac{10}{7} p_{1a} + p_{2d} \quad \wedge \quad p_{2c} + 10p_{3a} \geq 2p_{2d}$$

- Application: maximization of, e.g., p_{2d}
  - By instantiating all parameters except p_{2d} within K_0, we get $p_{2d} \leq \frac{34}{7}$
  - We can thus maximize p_{2d} to $\frac{34}{7}$ so that μ_0 remains optimal

Outline

Implementation

- IMPRATOR: program written in OCaml
  - IMPRATOR: "Inverse Method for Policy with Reward AbstracT BehaviOR"
  - 4000 lines of code
  - 2 man-months of work
- Features
  - Very intuitive input syntax
  - Solves the direct problem for (non-parametric) MDPs
  - Solves the inverse problem for parametric MDPs
- IMPRATOR will be available on its Web page
  - http://www.lsv.ens-cachan.fr/~andre/ImPrator
  - Coming (very) soon!

Outline

Final Remarks (1/2)

- Generalization method
  - Modeling of a system with a parametric Markov decision process $\mathcal{M}$
  - Starting with an instantiation π_0 of the parameters, as well a policy μ_0 optimal for $\mathcal{M}[\pi_0]$, we generate a constraint K_0 on the parameters guaranteeing that μ_0 is optimal for $\mathcal{M}[\pi]$, for any $\pi \models K_0$
- Advantages
  - Useful to optimize costs of systems, e.g., hardware devices
  - Powerful even on fully parametrized big systems
    - All case studies terminated in less than 1 second
- Applications
  - Real time systems
  - Hardware verification

Final Remarks (2/2)

- Other frameworks for the inverse method
  - Parametric Timed Automata [ACEF09]
    - Tool IMITATOR [And09]
  - Max–Plus Algebra [AF09]
    - Computation of the maximal circuit mean in a directed weighted graph
    - Tool under development

- Future works
  - Prove that the generated K_0 is maximal
    - If μ_0 is an optimal policy for $M[\pi]$, then $\pi \models K_0$
  - Handle MDPs with 2 kinds of weights
    - Example: (1) power consumption and (2) number of lost requests
    - Application: dynamic power management [PBBDM98]

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